# LARGE-SCALE LINEAR PROGRAMMING

BY

GEORGE B. DANTZIG

TECHNICAL REPORT NO. 67-8

NOVEMBER 1967



OPERATIONS RESEARCH HOUSE

Stanford University
CALIFORNIA

# LARGE-SCALE LINEAR PROGRAMMING

by

GEORGE B. DANTZIG

TECHNICAL REPORT NO. 67-8

NOVEMBER 1967

Department of Operations Research
Stanford University
Stanford, California

Research partially supported by Office of Naval Research, Contract ONR-N-00014-67-A-0112-0011; U.S. Atomic Energy Commission, Contract AT(04-3)-326 PA #18; National Science Foundation Grant GP 6431; and U.S. Army Research Office, Contract No. DAHC04-67-C-0028.

# LARGE-SCALE LINEAR PROGRAMMING

by George B. Dantzig*

## Large-Scale Systems and the Computer Revolution:

From its very inception, it was envisioned that linear programming would be applied to very large, detailed models of economic and military systems. Kantorovitch's 1939 proposals, which were before the advent of the electronic computer, mentioned such possibilities, [78]. Linear programming evolved out of the U.S. Air Force interest in 1947 in finding optimal time-staged deployment plans in case of war, [126]; a problem whose mathematical structure is similar to that of finding an optimal growth pattern of a developing economy and similar to other control problems, [41], [58], [123]. Structurally the dynamic problems are characterized in discrete form by staircase matrices representing the inputs and outputs from one time period to the next. Treated as an ordinary linear program, the number of rows and columns grows in proportion to the number of time periods $T$ and the computational effort grows by $T^3$ and possibly higher. This fact has limited the use of linear programming as a tool for planning over many time periods.

---

*Departments of Operations Research and Computer Science, Stanford University, Stanford, California. Research of G.B. Dantzig partially supported by Office of Naval Research, Contract ONR-N-00014-67-A-0112-0011; U.S. Atomic Energy Commission, Contract AT(04-3)-326 PA #18; National Science Foundation Grant GP 6431; and U.S. Army Research Office, Contract No. DAHC04-67-C-0028.

At the present 1967 stage of the computer revolution, there is growing interest on the part of practical users of linear programming models to solve larger and larger systems [40]. Such applications imply that eventually automated systems will obtain information from counters and sensing devices, process data into the proper form for optimization and finally implement the results by control devices. There has been steady progress in this mechanization of flow to and from the computer. Hitherto, this has been one of the obstacles encountered in setting-up and solving large-scale systems, [113]. The second obstacle has been the cost and the time required to successfully solve large problems, [74].

It is difficult to measure the potential of large-scale planning. Certain developing countries appear, according to optimal calculations on simplified models to be able to grow at the rate of 15% per year implying a doubling of their industrial base in 5 years. But administrators apparently ignore plans and make decisions based on political expediency which restrict growth to 2 or 3% and sometimes -2%. It is the belief of the author that the mechanization of data flow (at least in advanced countries) in the next decade will provide pathways for constructing large models and the effective implementation of the results of optimization. This application of mathematics to decision processes will eventually become as important as the classical applications to physics and will, in time, change the emphasis in pure mathematics.

Three Approaches to Solving Large-Scale Systems:

There have been a great number of papers on this subject as evidenced from the list of references attached. I have broadly classified them into:

| | |
|---|---|
| I | Decomposition Principle<br>(Sub-optimization using interior path) |
| II | Compact Inverse<br>(Using a simplex variant) |
| III | Parametric Variation<br>(Sub-optimization using simplex variant) |

The aim is to say a little about each, citing some references and some structures to which they are applicable. We shall begin with

I: The Decomposition Principle, [47]:

Consider the non-linear programming problem: Find $x = (x_1,\ldots,x_n)$ such that

$$
\begin{aligned}
(1)\qquad & g(x) = \text{Min} \\
& f_1(x) \leq 0 \qquad :\lambda_1 \\
& f_2(x) \leq 0 \qquad :\lambda_2 \\
& f_3(x) \leq 0 \\
& f_m(x) \leq 0
\end{aligned}
$$

We assume $g(x)$ and $f_i(x)$ are convex functions of $x$. Assigning Lagrange Multipliers $\lambda_i$ to a subset of the constraints, say the first two, we obtain the SUBPROBLEM: Find $x$ and Min $\emptyset(x)$ satisfying

$$
\begin{aligned}
(2)\qquad & \emptyset(x) = g(x) + \lambda_1 f_1(x) + \lambda_2 f_2(x) \\
& f_3(x) \leq 0,\ldots,f_m(x) \leq 0
\end{aligned}
$$

Theorem: If for given $\lambda_i$ , $x = \hat{x}$ solves the subproblem (2) and if $f_i(\hat{x}) \leq 0$ for all i and $\lambda_i f_i(\hat{x}) = 0$ for i = (1,2) then $x = \hat{x}$ solves (1).

We shall discuss a method where we assign values to $\lambda_i$ and if the resulting $x = \hat{x}$ does not satisfy the conditions in the theorem, this fact can be used to improve the values of $\lambda_i$ .

(Ia) Example:

FIND $x \geq 0$ , Min $f_o(x)$:

$$
\begin{array}{llllllll}
(3) & c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4 + c_5 x_5 & & & & = f_o(x) & & \\
 & a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + a_{15}x_5 & & & & = b_1 & : \lambda = \lambda_1 & \boxed{\text{GUESS}} \\
 & a_{21}x_1 + a_{22}x_2 & & & & \leq b_2 & & \\
 & a_{31}x_1 + a_{32}x_2 & & & & \leq b_3 & & \\
 & & a_{43}x_3 + a_{44}x_4 & & & \leq b_4 & & \\
 & & a_{53}x_3 + a_{54}x_4 & & & \leq b_5 & & \\
 & & & & a_{65}x_5 & \leq b_6 & & \\
 & \multicolumn{7}{l}{- - - - - - - - - - - - -} \\
 & \multicolumn{5}{l}{\phi_1(x_1,x_2) + \phi_2(x_3,x_4) + \phi_3(x_5)} & = \phi(x) \text{ Min} & \text{SUBPROBLEM}
\end{array}
$$

where $\phi_1 = (c_1 + \lambda a_{11})x_1 + (c_2 + \lambda a_{12})x_2$; $\phi_2 = (c_3 + \lambda a_{13})x_3 + (c_4 + \lambda a_{14})x_4$;

$\phi_3 = (c_5 + \lambda a_{15})x_5$

Note that the subproblem decomposes into three separate problems; hence the term: "Decomposition Principle".

(Ib) Equivalent Generalized Linear Program:

Returning to problem (1) we now restate it in the form of

Wolfe's Generalized Linear Program, [38, Chapter 22]. This differs from an ordinary linear program in that the coefficients in each column $P_j$ instead of being fixed are freely drawn from a convex set $C_j$. It can be shown that the following problem is equivalent to (1).

FIND Min $z$, $w_i \geq 0$ such that

$$(4)\quad \underbrace{\begin{bmatrix}0\\0\\0\\1\end{bmatrix} \geq \begin{bmatrix}-1\\0\\0\\0\end{bmatrix} z + \begin{bmatrix}g(x^1)\\f_1(x^1)\\f_2(x^1)\\1\end{bmatrix} w_1 + \ldots + \begin{bmatrix}g(x^t)\\f_1(x^t)\\f_2(x^t)\\1\end{bmatrix} w_t}_{\text{RESTRICTED MASTER}} + \begin{bmatrix}g(\bar{x})\\f_1(\bar{x})\\f_2(\bar{x})\\1\end{bmatrix} w \qquad \begin{matrix}: 1\\: \lambda_1\\: \lambda_2\\: \mu\end{matrix}$$

where $x^i$ and $\bar{x}$ satisfy $f_3(x) \leq 0, \ldots, f_m(x) \leq 0$ and the solution to (1) is

$$(5)\qquad \hat{x} = \sum w_i x^i + w\bar{x}.$$

(Ic) Iterative Process:

At the start of iteration $t$, $x^1, \ldots, x^t$ are known. An improved guess of $(\lambda_1, \lambda_2)$ and a new $x = x^{t+1}$ is obtained by solving the "restricted master" linear programming indicated in (4). Let the optimal dual variables be $(1, \lambda_1^t, \lambda_2^t, \mu^t)$ and let $w_i = w_i^t$ be the optimal primal variables. Then

$$(6)\qquad \hat{x}^t = \sum_{i=1}^{t} w_i^t x^i$$

is an optimal solution to (1) if

$$(7)\quad \text{Min } [g(\bar{x}) + \lambda_1^t f_1(\bar{x}) + \lambda_2^t f_2(\bar{x}) + \mu] \geq 0,$$
$$f_1(\bar{x}) \leq 0, \qquad f_2(\bar{x}) \leq 0;$$

i.e. if the last column "prices out" non-negative for all admissable $\bar{x}$ .
But (7) is the same as solving the subproblem (2) using $(\lambda_1, \lambda_2) = (\lambda_1^t, \lambda_2^t)$ .
If in (7), $x = x^{t+1}$ yields a Min < 0 , this $x$ is used to generate a new column of (4).

The successive $\hat{x}^t$ satisfy $f_i(x) \leq 0$ for all $i$ and $g(\hat{x}^t) \to \text{Min } g(x)$ . The iterative process is finite when applied to a linear program like the preceeding example (3).

This completes our discussion of the decomposition approach. To be useful, the generated subproblems must be <u>easy</u> to solve, [38,Chapter 24].

<u>(II) Compact Inverse</u>:

The second approach accepts the standard simplex or any of the numerous variants and tries to arrange the arithmetic to take advantage of structure. It is clear that if the number of iterations is fixed, the only savings can come from doing each iteration efficiently: i.e. doing the pricing and those operations involving the inverse of the basis efficiently.

<u>(IIa) Sparse Matrices</u>:

The larger problems become the lower, in practice, become the density of non-zero coefficients. For problems of 200 equations a density of 5% is typical; for larger problems the density drops to .5% or less. It is possible, however, that the inverses of bases drawn from such matrices to be 100% dense, for example:

$$(8)\quad B = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 1 & & \\ 1 & & 1 & \\ 1 & & & 1 \end{bmatrix};\ B^{-1} = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 2 & 1 & 1 \\ -1 & 1 & 2 & 1 \\ -1 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ -1 & & 1 & \\ -1 & & & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & -1 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

However, if the inverse is expressed as products of elementary matrices of either the row or column type or both in any order, the number of off-diagonal non-zeros in this representation can often be quite low.

Unsolved problem: Given a basis express the inverse as the product of elementary matrices such that the count of off-diagonal non-zero elements is minimal.

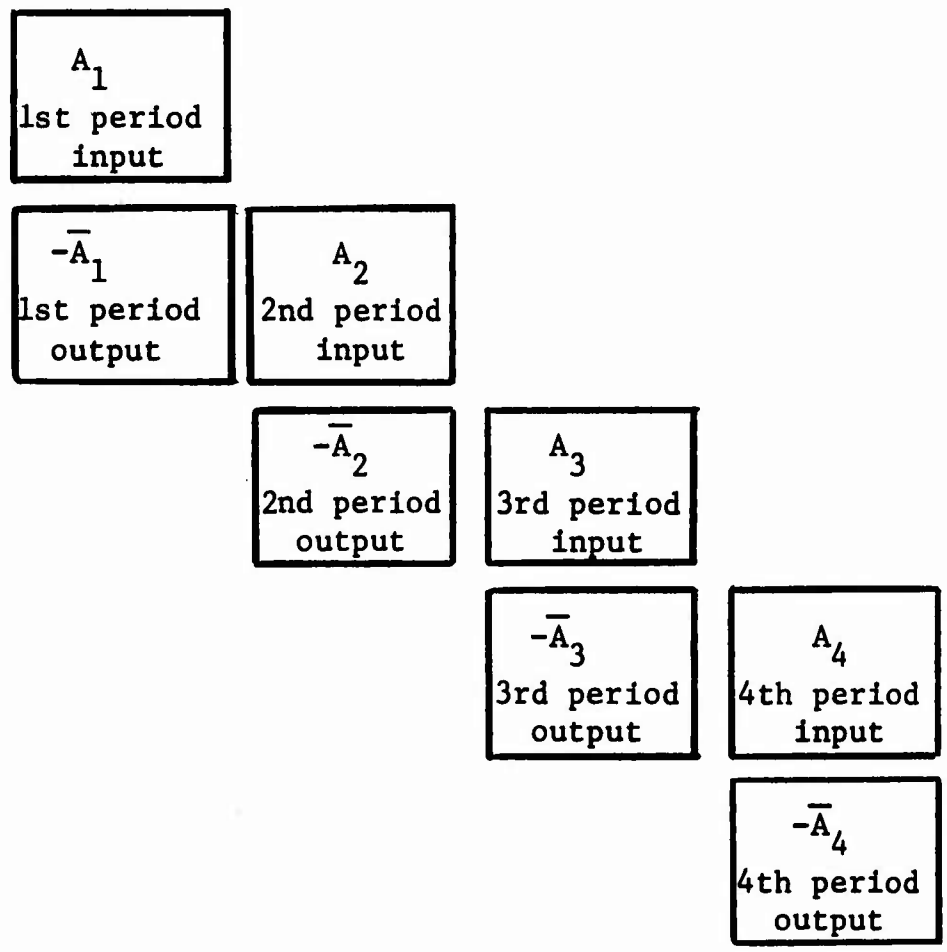
Markowitz [90] proposed that the elementary matrices correspond to upper and lower diagonalization operations using as pivot element the one that locally creates as few additional non-zeros as possible. Variations of this idea have been incorporated in commercial codes in the early 1960's, see [43]. The inverse of a 5% dense basis often running not more than 7% dense and the running time often is cut by a factor of 5. In example (8), the inverse in product form has the same number of non-zeros as the originating basis.

(IIb) Dynamic Structures:

As noted earlier these have important applications [95]. One such is to linear control processes, see [114], [128]. As early as 1954, the author published a paper on how to compact the inverse representation of the basis with a staircase structure, (9); see [32].

Again, in 1963, I discussed another method which also permitted one to find a compact inverse and efficiently maintain the compactness in moving from one iteration to the next, [37]. There have been other proposals, all excellent, that seek to apply the simplex method to the full system by compacting the inverse. As far as I know, none of these <u>direct</u> proposals have been realized in computer codes. See [5], [56], [71].

(9)




An important special case is the <u>Dynamic Leontief Economic Model with Substitution</u> [33]. Another Special Case is a <u>Markov Process with Alternative Policies</u> [125], [76]. These cases are known to be mathematically equivalent and to have a remarkably simple solution. A Leontief System is defined by: (1) a non-negative right hand side,

(2) exactly one positive coefficient in each column, and (3) the existence of a feasible solution for some positive right hand side. In the dynamic case, we further assume that the positive coefficient always appears in the input block along the diagonal.

Theorem: The optimal choice of basic columns associated with the last period is independent of the choice in prior periods.

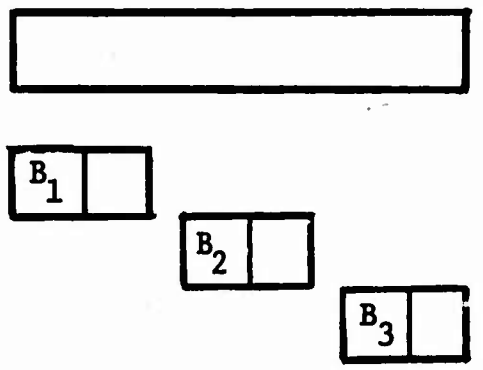
This permits the determining of optimal basis and Lagrange multipliers for the last block of equations. Weighting these equations by their multipliers, the last period equations are subtracted from the cost equations to produce a modified-cost equation. Dropping these equations, the optimal choice of columns for the next-to-last period and prices are next determined using the modified-cost equation; etc. backwards in time until the first period is reached. When the basic columns of the first period become known, the value of its basic variables can be calculated, these in turn can be used to determine those of the second period, etc. forward in time. [118].

The essential characteristic of the basis in the dynamic Leontief case and in the Markov Process case is that the blocks of non-zero coefficients are square and non-singular and the entire basis is block triangular. Hence only the inverses of blocks along the diagonal are needed; the rest of the calculations can be done by substitution below the diagonal. An ideal block-triangular structure! Unfortunately, the general staircase problem does not have this property. It would be very worthwhile to see if one can find a meaningful economic extension of

the Leontief model (like the introduction of activities that generate capital) that is tractable.

(IIc) Block Angular Systems: These consist of M general linear equations and L sets of equations which have no variable in common. The blocks of non-zero coefficients are depicted below.

(10)




Several proposals have been made to compact the inverse, see particularly Bennett [21]; also [79], [106]. Essentially they all chose square non-singular submatrices $B_i$ from the basis along the block diagonal which are used as block pivots to initiate the elimination. After the elimination, a square m x m submatrix is left. Many practical problems satisfy this structure. One important subclass are the multi-commodity network problems, [54], [77], sometimes referred to as the traffic assignment problem [24]:

(11) Find $x_{ijk} \geq 0$ , Min z :

$$\sum_k x_{ijk} \leq c_{ij} \qquad (i,j,k) = 1,\ldots,n.$$

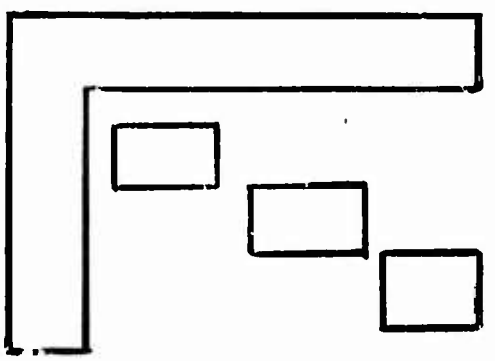
$$\sum_i x_{ipk} - \sum_j x_{pjk} = a_{pk} \qquad (p,k) = 1,\ldots,n$$

$$\sum_i \sum_j \sum_k t_{ijk}\, x_{ijk} = z$$

In another type of application involving the allocation of many orders to several plants, the diagonal blocks consist of one equation each. Such a system is referred to as a <u>generalized upper-bound structure</u>, [46]. In one application $L = 4000$ and $M = 20$. An important property of such systems is that when $L$ is large relative to $M$ most (in fact $L-M$ or more) of the diagonal equations have exactly one basic variable among the set of its variables. The fact that most basic variables are at their upper-bound value can be used to advantage. The first code along these lines was developed by M. Kasatkin and J. Bigelow for a problem of Crown Zellerbach paper corporation. Running time on an example was about 1/10 the time that was required by a general purpose code. See also [65].

<u>(IId) Bordered Angular Systems</u>: This consists of blocks along the diagonal of non-zero coefficients and a border of non-zeros along the top and left.

(12)



This structure is sufficiently general yet specialized to usefully cover

a majority of current applications except the staircase type. Generalization (of the procedures just discussed) have been made by Heesterman, [72]. Ritter [99] has generalized Rosen's parametric scheme, [103].

III. Parametric Variation:

The third and last approach depends on the system being weakly linked i.e. on the existence of a few rows and columns which, if removed, makes the solution of the remaining system trivial. For example, a network-flow problem with an extra budget constraint. By assigning a Lagrange Multiplier to the latter, the constraint could be removed and the objective equation modified by adding to it the multiple of the removed constraint. The resulting pure network could then be easily solved. If the solution does not satisfy the constraint and complementary slackness conditions, then the Lagrange Multiplier is varied until it does. This is also the idea behind the decomposition principle but the proposed methods of variation (such as those below) are more direct:

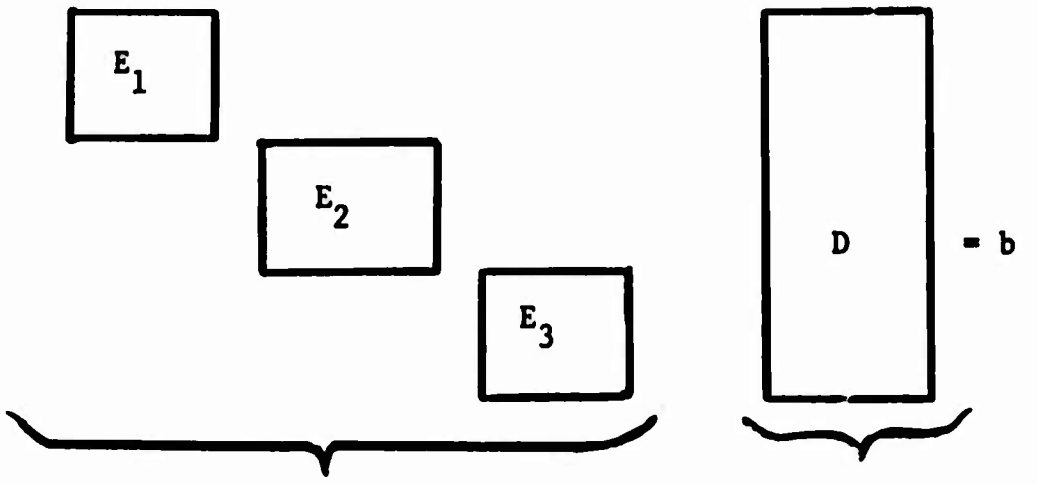
Rosen: "Partition Programming" [103], Ritter [99].
Kron: "Diakoptics" [83].
Balas: "Infeasibility Pricing" [10].
Beale: "Pseudo Basic Variables" [17].
Abadie & Williams: [3].
Gass: "Dualplex Method" [59].

(IIIa) Dualplex Method:

As representative of the parametric approaches I have selected

Gass' "Dualplex Method" which is related to Rosen's "Partition Programming" in dual form. It is clear if we had a <u>transposed block-angular</u> structure

(13)




that pivoting in the right hand interconnecting part would destroy the angular structure but pivoting anywhere in $E_1$, $E_2$, $E_3$ would not. We assume that for a given $Y = Y^o \geq 0$ (variables associated with $D$) a feasible solution $X = X^o \geq 0$ exists and is optimal. Let the system be reduced to optimal canonical form restricting pivots to only columns of $E_i$ :

$$\begin{aligned} IX_B + \bar{A}X_N + \bar{D}Y &= \bar{b} \\ \bar{c}X_N + \bar{d}Y &= z - Z_o \ (\text{Max}) \end{aligned} \tag{14}$$

where $X_B$ are basic variables and $X_N, Y$ non-basic. Holding $X_N = 0$ for the moment, we solve the subproblem

$$\bar{D}Y \leq \bar{b} \ , \quad Y \geq 0 \ , \quad \text{Max} \ \bar{d}Y \ . \tag{15}$$

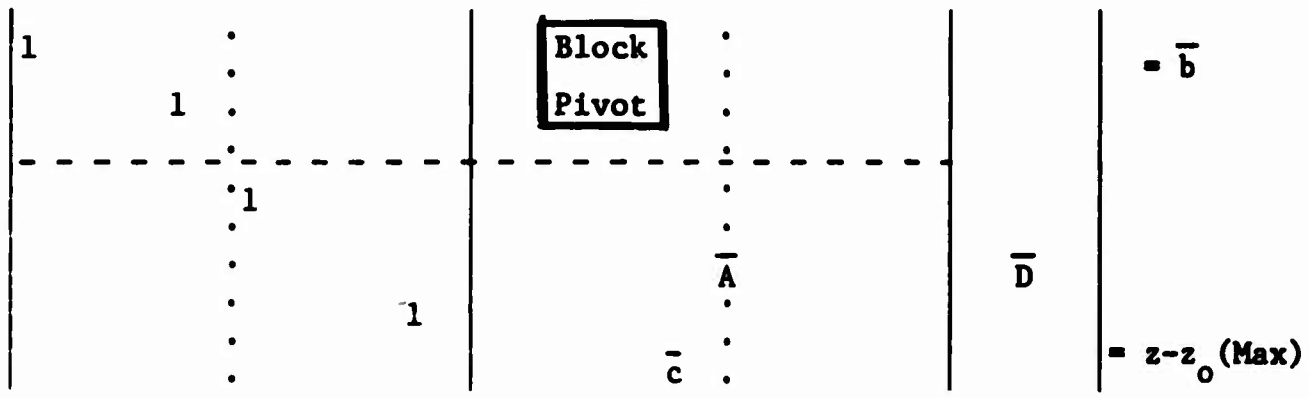
The dual of this subproblem is

$$\Pi\bar{D} \geq \bar{d}\ , \quad \Pi \geq 0\ , \quad \text{Min}\ \ \Pi\bar{b}\ . \tag{16}$$

Since $\bar{D}^T$ is presumed to consist of few rows and many columns, it is suitable for solution by the standard simplex method. Let $\Pi = \Pi^1$ be an optimal solution and $Y = Y^1 \geq 0$ be optimal to its dual. Denote by $\bar{D}_i$ the i-th row of $\bar{D}$ and by $\bar{A}_j$ the j-th column of $A$. Let the basic $X_B$ be partitioned into $X_I = 0$ and $X_{II} > 0$ according as components $x_i = 0$ or $x_i > 0$ where $x_i + \bar{D}_i Y' = \bar{b}_i$ ; Let the non-basic $X_N$ be partitioned into $X_{III}$ and $X_{IV}$ according as $\delta_j = \bar{c}_j - \Pi\bar{A}_j > 0$ or $\leq 0$.

$$\delta_{III} > 0 \qquad \delta_{IV} \leq 0$$

(17)




The block pivot:

The next step is to find the block pivot of highest rank that switches the role of as many basic and nonbasic variables in $X_I$ and $X_{III}$ as possible. Since both sets are at zero value this does not effect the current feasible solution. If there is a choice of block pivot its columns are selected from those with highest $\delta_j$ values.

After the pivot the new dual subproblem is solved using as starting basis, the one corresponding to the final basis of the previous subproblem. $Y' \geq 0$ is still a feasible price vector of the dual subproblem but $\Pi'$ no longer satisfies it. However,

Theorem (Gass):

<u>If after the block pivot those components $\Pi'_j$ of $\Pi'$ corresponding to $\delta_i > 0$ are replaced by the value $-\delta_i$, the new $\Pi$ constitutes an infeasible basic solution to the new subproblem; $Y' \geq 0$ remains as a feasible vector of dual simplex multipliers.</u>

Because of infeasibility, the new subproblem can be improved (using the dual simplex method). This is repeated iteratively until all $\delta_j \leq 0$ or $z \to +\infty$. Associated with each iteration is a basic feasible to the full problem so that usual proof of a finite-number-of-iterations applies.

The parametric methods should be regarded as important variants of the standard simplex process.

<u>Concluding Remarks</u>:

This completes the survey of the three types of approaches to solving large-scale systems: Decomposition, Compact Inverse, and Parametric Variation, and of the type of matrix structures that each are best suited. Little has been said about how different proposals compare on test problems. At present, there does not appear to be a

practical way to do this. The program of instructions for the computer are often an order of magnitude more complex than a good commercial linear program system and the latter can cost two to five hundred thousand dollars to develop. The author feels that better computer languages have to be developed to facilitate the experimental coding and comparision of large-scale system proposals, [74].

## BIBLIOGRAPHY ON LARGE-SCALE SYSTEMS


1. ABADIE, J.M., "Dual Decomposition Method for Linear Programs", Comp. Center Case Institute of Technology, July 1962.

2. ABADIE, J.M., "On Decomposition Principle", Operations Research Center, University of California, Berkeley, ORC 63-20, 1963.

3. ABADIE, J.M. and WILLIAMS, A.C., "Dual and Parametric Methods in Decomposition", in Recent Advances in Math. Prog., edited by R. Graves and P. Wolfe, McGraw-Hill, 1963.

4. ACZEL, M.A. and RUSSEL, A.H.,"New Methods of Solving Linear Programs", O.R. Qu. Vol. 8 No. 4, Dec. 1957.

5. ADIN, B. Thomas, "Optimizing a Multistage Production Process", O.R. Qu. Vol. 14, No. 2, June 1963.

6. AGGARWAL, S.P., "A Simplex Technique for a Particular Convex Programming Problem", Canadian Operational Research Journal, Vol. 4, No. 2 July 1966.

7. ALTMAN, M., "An Elimination Method for L.P. with Application to the Decomposition Problem", Bull. Acad. Polon. Sci. Ser. Sci. Math. Astron. Physics.

8. ALWAY, G.G., "A Triangularization Method for Computations in Linear Programming", Naval Research Logistics Quarterly, Vol. 9, pp. 163-180.

9. BAKES, M.D., "Solution of Special Linear Programming Problem with Additional Constraints", O.R. Qu. Vol. 17, No. 4, Dec. 1966.

10. BALAS, Egon, "An Infeasibility - Pricing Decomposition Method for Linear Program", July 1966, Operations Research 14 (1966) 843-873.

11. BALAS, Egon, :Solution of Large Scale Transportation Problems Through Aggregation", Operations Research, 13 (1965) 82-93.

12. BALINSKI, M.L., "On Some Decomposition Approaches in Linear Programming", and "Integer Programming", The University of Michigan Engineering Summer Conferences, 1966.

13. BARNETT, S., "Stability of the Solution to a Linear Programming Problem", O.R. Qu., Vol. 13, No. 3, September 1962.

14. BAUMOL, W.J. and FABIAN, T., "Decomposition, Pricing for Decentralization and External Economics", Management Science, Vol. 11 No. 1, September 1964.

15. BEALE, E.M.L., "Survey of Integer Programming", O.R. Qu. Vol. 16, No. 2, June 1965.

16. BEALE, E.M.L., "Decomposition and Partitioning Methods for Nonlinear Programming", in Non-Linear Programming, J. Abadie, Ed., North-Holland publishing Company, also Wiley.

17. BEALE, E.M.L., "The Simplex Method Using Pseudo-Basic Variables for Structured Linear Programming Problems", from Recent Advances in Math. Prog., edited by R. Graves and P. Wolfe, McGraw-Hill, 1963.

18. BELL, E.J., "Primal-Dual Decomposition Programming", Unpublished Ph.D. Thesis, Industrial Engineering Department, University of California, Berkeley, 1964.

19. BELLAR, F.J., "Iterative Solution of Large-Scale Systems of Simultaneous Linear Equations", SIAM Journal, Vol. 9, No. 2, June 1961.

20. BENDERS, J.F., "Partitioning Procedures for Solving Mixed Variables Programming Problems", Num. Math. 4, 1962.

21. BENNETT, J.M., "An Approach to Some Structured Linear Programming Problems" Operations Research 14 (1966) 4 (July-August) pp. 636-645.

22. BESSIERE, F. et SAUTER, E., "Optimisation et Enviornment Economique: La Methode Des Modeles Flargis", Revue Francaise de Recherche Operationnelle No. 40, 1966.

23. BOOT, J.C.G., "On Trivial and Binding Constraints in Programming Problems", Management Sci. Vol. 8, 1962, pp. 419-441.

24. BRADLEY, S.P., "Solution Techniques for the Traffic Assignment Problem", ORC 65-35, University of California, Berkeley, 1965.

25. BRASILOW, C.B., LASDON, L.S., PEARENS, J.D., MACKO, O., TAKAHORA, Y., "Papers on Multilevel Control Systems", DTV 70-A-65, Case Institute of Technology, 1965.

26. CATCHPOLE, A.R., "The Application of Linear Programming to Integrated Supply Problems in the Oil Industry", O.R. Qu., Vol. 13, No. 2, June, 1962.

27. CHARNES, A. and COOPER, W.W., "Generalizations of the Warehousing Model", O.R. Qu. Vol. 6, No. 4, Dec. 1955.

28. CHARNES, A. and COOPER, W.W., "Management Models and Industrial Applications in Linear Programming", Management Science, Vol. 4, No. 1 October 1957, pp. 38-91.

29. CHURCHMAN, C.W., "On the Ethics of Large-Scale Systems, Part I", Internal Working Paper, No. 37, SSL, University of California, Berkeley, September 1965.

30. CRAVEN, B.D., "A Generalization of the Transportation Method of Linear Programming", O.R. Qu. Vol. 14, No. 2, June 1963.

31. CURTES, H.A., "Use of Decomposition Theory in the Solution of the State Assignment Problem of Sequential Machines", Journal of A.C.M., July 1963, p. 386.

32. DANTZIG, G.B., "Upper Bounds, Secondary Constraints and Block Triangularity in Linear Programming", Econometrica, Vol. 23, No. 2 April, 1955.

33. DANTZIG, G.B., "Optimal Solution of a Dynamic Leontief Model with Substitution", Econometrica, Vol. 23, No. 3, July 1955.

34. DANTZIG, G.B., "Linear Programming Under Unvertainty", Management Science, Vol. 1 (1955) pp. 197-206.

35. DANTZIG, G.B., "On the Status of Multistage Linear Program", The RAND Corp. p. 1028, 20 Feb. 1957, Proc. International Statistical Institute, Stockholm, 1957.

36. DANTZIG, G.B., "On the Status of Multistage Linear Programming Problems", Management Science, Vol. 6, No. 1, October 1959, Also in, Mathematical Studies in Management Science, - Veinott.

37. DANTZIG, G.B., "Compact Basis Triangularization for the Simplex Method", from Recent Advances in Math. Prog. edited by R. Graves and P. Wolfe, McGraw-Hill, 1963.

38. DANTZIG, G.B., "Linear Programming and Extensions" Princeton University Press, 1963, 1966.

39. DANTZIG, G.B., "Large Scale System Optimization", ORC 65-9, University of California, Berkeley, 1965.

40. DANTZIG, G.B., "Operations Research in the World of Today and Tomorrow", Operations Research Center, 1965-67, University of California, Berkeley; also in Management Science, January 1965.

41. DANTZIG, G.B., "Linear Control Processes and Mathematical Programming, SIAM Journal, Vol. 4, No. 1, 1966.

42. DANTZIG, G.B., FULKERSON, D.R., and JOHNSON, S., "Solution of a Large Scale Travelling Salesman Problem", JORSA, Vol. 2, No. 4, November 1954, p. 393.

43. DANTZIG, G.B., HARVEY, R., McKNIGHT, R., "Updating the Product Form of the Inverse for the Revised Simplex Method", Operations Research Center 1964-33, University of California, Berkeley.

44. DANTZIG, G.B., and MADANSKY, A., "On the Solution of Two-Stage Linear Programs under Uncertainty", Proceedings, Fourth Symposium on Mathematical Statistics and Probability, Vol. 1, 1961, pp. 165-176.

45. DANTZIG, G.B., and HAYS, W. Orchard, "Alternative Algorithm for the Revised Simplex Method using Product Form for the Inverse." The RAND Corp. RM-1268, 1953.

46. DANTZIG, G.B., and VAN SLYKE, R.M., "Generalized Upper Bounded Techniques for Linear Programming, I, II", Operations Research Center, University of California, Berkeley, ORC 64-17,18; also in Proceedings of the IBM Scientific Computing Symposium on Combinatorial Problems, March 16-18, 1964, pp. 249-261, and Journal of Computer and System Sciences, issue 2 forthcoming.

47. DANTZIG, G.B., and WOLFE, P., "The Decomposition Algorithm for Linear Programming", Econometrica, Vol. 29, No. 4, October 1961, Operations Research, Vol. 8, No. 1, January, February, October 1960.

48. DENNIS, D.E. (abstract) "A Multi-Period Transportation Problem", Econometrica, Vol. 31, 1963, p. 595.

49. DZIELINSKI, P. and GOMORY, R.E. (abstract) "Lot Size Programming and the Decomposition Principle" Econometrica, Vol. 31, 1963, p. 595.

50. EL AGIZY, M., "Programming Under Uncertainty with Discrete D.F.". ORC 64-13, University of California, Berkeley, July 1964, Ph.D. Thesis.

51. ELMAGHRABY, S.E., "An Approach to L.P. under Uncertainty", JORSA Vol. 7, No. 2, March, April 1959, p. 208.

52. ELECTRICITE DE FRANCE, "Programmes Lineaires Method de Decomposition", Direction de Etudes et Recherches, Paris, June 13, 1961.

53. ELECTRICITE DE FRANCE, "Programmation Lineaire, Methode de Dantzig et Wolfe, Programme Experimental", Direction des Etudes et Recherches, Paris, May 28th, 1962.

54. FORD, Lester, Jr., FULKERSON, D.R., "Suggested Computation for Maximal Multi-Commodity Network Flows", the RAND Corp., R-1114, Management Science, Oct. 1958, Vol. 5, No. 1.

55. FRISCH, R., "Tentative Formulation of the Multiplex Method for the Case of a Large Number of Basic Variables", Institute of Economics, University of Oslo, March 1962.

56. FULKERSON, D.R., "A Feasibility Criterion for Staircase Transportation Problems and Application to a Scheduling Problem" The RAND Corp., Report P. 1188, October 1957.

57. FAURE, P. et HUARD, P., "Résolution de Programmes Mathématiques à Fonction non Lineaire par la Mêthode due Gradient Rêduit", No. 36, 1965, Revue Francaise de Recherche Operationnelle.

58. GALE, David, "On Optimal Development in a Multi-Sector Economy", Operations Research Center, 1966-11, University of California, Berkeley, April 1966.

59. GASS, Saul I., "The Dualplex Method for Large-Scale Linear Programs", Operations Research Center, 1966-15, University of California, Berkeley, June 1966, Ph.D. Thesis.

60. GAUTHIER, J.M., "Le principle de Decomposition de Dantzig et Wolfe", Groupe de Travail, Mathematiques de Programmes Economiques, March 13, 1961.

61. GEOFFRION, A.M., "Direct Reduction of Large Concave Programs" Working Paper III, WMSI, UCLA, December 1966.

62. GILMORE, P.C. and GOMORY, R.E., "The Theory and Computation of Knapsack Functions", Operations Research, Vol. 14, No. 6, Nov-Dec. 1966.

63. GOMORY, R.E., "Large and Non-Convex Problems in Linear Programming" RC-765, IBM, August, 1962; see also [64].

64. GOMORY, R.E., "Large and Non-Convex Problems in L.P.", Proc. Sympos. Appl. Math. 15 (1963) 125-139.

65. GOMORY, R.E., and HU, T.C., "An Application of Generalized Linear Programming to Network Flows", IBM Research Report (1960) 50 p., SIAM Journal Vol. 10, No. 2, June 1962.

66. GOULD, S., "A Method of Dealing with Certain Non-Linear Allocation Problems Using the Transportation Technique", Operations Research Qu. Vol. 10, No. 3, September 1959.

67. GRAVES, R.L., WOLFE, P. (eds.) Recent Advances in Mathematical Programming, (McGraw-Hill Book Co., New York, 1963, 347 pp.)

68. HADLEY, G., Linear Programming, p. 437-508.

69. HALEY, R.B., "A General Method of Solution for Special Structure Linear Programmes", O.R. Qu. Vol. 17, No. 1, March 1966.

70. HARVEY, R.P., "Decomposition Principle for Linear Programming", Int. Jour. Comp. Math. May 1964 (20-35).

71. HEESTERMAN, A.R.G., "Partitioning a Phased Linear Programming Problem", Central Plan Bureau, Stolkweg, Working paper, April 1962.

72. HEESTERMAN, A.R.G., "Special Simplex Algorithm for Multi-Sector Problems", Series A #68, University of Birmingham, 1965.

73. HEESTERMAN, A.R.G., SANDEA, J., "Special Simplex Algorithm for Linked Problems", Management Science, 11,3 (January 1965) 420-428.

74. HELLERMAN, Eli, "The Dantzig-Wolfe Decomposition Algorithm as Implemented on a Large-Scale (Systems Engineering) Computer", Presented at Modern Techniques in the Analysis of Large-Scale Engineering Systems, Nov. 1965.

75. HERSHKOWITZ, M., and NOBLE, S.B., "Finding the Inverse and Connections of a Type of Large Sparse Matrix", Naval Research Logistics Quarterly, Vol. 12, No. 1, pp. 119-133.

76. HITCHCOCK, D.F., and MacQUEEN, J.B., "On Computing the Expected Discounted Return in a Markov Chain", Working Paper No. 105, Western Management Science Institute, UCLA, August 1966.

77. HU, T.C., "Multi-Commodity Network Flow", IBM Watson Research Center, Research Report RC-865, January 1963, and Operations Research, Vol. 11, 1963, pp. 344-360.

78. KANTOROVITCH, L.V. "Mathematical Methods in Organization and Planning of Production", Leningrad 1939, translated in Management Science, Vol. 6, 1960, pp. 366-422.

79. KAUL, R.N., "An Extension of Generalized Upper-Bounded Techniques for Linear Programming", Operations Research Center, University of California 1965-27, Berkeley, August 1965.

80. KLEE, Victor, "A Class of Linear Programming Problems Requiring a Large Number of Iterations", Numerische Mathematik, 7,313-321 (1965).

81. KRON, G., "Piecewise Solution of Large Scale Systems", General Electric, July, 1957.

82. KRON, G., "Piecewise Optimization of Linear Programming", General Electric, December 1958.

83. KRON, G., DIAKOPTICS - The Piecewise Solution of Large-Scale Systems, Macdonald Publishers 2 Portman, St., London W1.

84. KUNZI, H.P., and TAN, S.T., "Lineare Optimierung Grozer Systeme", Springer-Verlag, Berlin, 1966.

85. LABRO, C., "Efficiency and Degrees of Decomposition", Working Paper No. 98, Centre for Research in Management Science, University of California, Berkeley, August 1964.

86. LANCZOS, C., "Iterative Solution of Large-Scale Linear Systems", SIAM Journal, Vol. 13, No. 1, March 1958.

87. LAND, A.H., "A Problem of Assignment with Inter-related Costs", O.R. Qu., Vol. 14, No. 2, June 1963.

88. MACGUIRE, C.B., "Some Extensions of the Dantzig-Wolfe Decomposition Scheme", Center for Research in Management Science, University of California, Berkeley, Working Paper, No. 66, March 1963.

89. MALINVAUD, E., "Decentralized Procedures for Planning", Technical Report No. 15, Center for Research in Management Science, University of California, Berkeley, 1963.

90. MARKOWITZ, "The Elimination Form of the Inverse and its Application to Linear Programming", The Rand Corp. p. 680, 1955.

91. MURTY, K.G., "Two-Stage Linear Programming Under Uncertainty: A Basic Property of the Optimal Solution", O.R. Center 1966-4, University of California, Berkeley, February, 1966.

92. NASLUND and WHINSTON, "Model for Multi-Period Decision Making Under Uncertainty", M.S. 8 (1962).

93. NEMHAUSER, G.L., "Decomposition of Linear Programming by Dynamic Programming", Naval Research Logistics Quarterly, Vol. 11, June-September 1964, pp. 191-196.

94. PARIKH, S.C. and JEWELL, W.S., "Decomposition of Project Networks", Management Science, Vol. 11, No. 3, January, 1965.

95. PARIKH, S.C., :Linear Dynamic Decomposition Programming of Optimal Long Range Operations of a Multiple Multi-Purpose Reservoir System", Presented at Fourth International Conference on O.R. 1966.

96. PEARSON, J.D., "Duality and a Decomposition Technique", SIAM Journal, Vol. 4, No. 1, 1966.

97. RECH, Paul, "Optimization by Price Communication between Leontief Expressions", O.R. Center 1964-35, University of California, Berkeley, December 1964.

98. RECH, Paul, "Decomposition and Interconnected Systems in Mathematical Programming", O.R. Center 1965-31, University of California, Berkeley, September, 1965, Ph.D. Thesis.

99. RITTER, K., "A Decomposition Method for Linear Programming Problems with Coupling Constraints and Variables", Math. Research Center, The University of Wisconsin #739, April 1967.

100. ROBERTS, J.E., "A Method of Solving a Particular Type of Very Large Linear Programming Problem", Proceedings of the First Annual Conference, Canadian Operational Research Society, University of Toronto, May 7-8, 1959, pp. 25-26.

101. ROBERTS, J.E., "A Method of Solving a Particular Type of Very Large Linear Programming Problem", Canadian Operational Research Journal, Vol. 1, No. 1. December 1963.

102. ROSEN, J.B., "Partition Programming", Notices, American Math. Soc. Vol. 7, 1960, pp. 718-719.

103. ROSEN, J.B., "Primal Partition Programming for Block Diagonal Matrices", Computer Science Division, School of Humanities and Sciences, Stanford University, Technical Report No. 32, November 1963; Numerische Math. 6,3(1964), 250-264.

104. ROSEN, J.B. and ORNEA, J.C., "Solution of Nonlinear Programming Problems by Partitioning", Shell Development Company, Emeryville, California, p-1115, June 1962.

105. SAIGAL, Romesh, "Block - Triangularization of Multi-Stage Linear Programs", O.R. Center 1966-9, University of California, Berkeley, March, 1966.

106. SAIGAL and SAKAROVITCH, "Compact Basis Triangularization for the Block Angular Structures", ORC, University of California, Berkeley (66-1).

107. SANDERS, J.L., "A Non-linear Decomposition Principle", Operations Research, 13 (1965), 266-67.

108. SHETTY, C.M., "On Analyses of the Solution to a Linear Programming Problem", O.R. Qu. Vol. 12, No. 2, June 1961.

109. SINHA, "Stochastic Programming, ORC, University of California, Berkeley, 63-22.

110. SIMONNARD, "Programmation Lineaire", Dunod (Paris) 1962.

111. STEINBERG, N., "Le Problème de Transport Généralisé à n dimensions, avec Données Aliatoires", Revue Francaise de Recherche Operationnelle, No. 26, 1963.

112. TCHENG, Tse-Hao, "Scheduling of a Large Forestry-Cutting Problem by Linear Programming Decomposition", Ph.D. Thesis, University of Iowa, August 1966.

113. THOMPSON, P.M., "Editing Large Linear Programming Matrices", JORSA, Vol. 4, No. 1, March 1957, pp. 97-100.

114. VAN SLYKE, R., "Mathematical Programming and Optimal Control", 1964, Ph.D. Thesis, University of California, Berkeley.

115. VAN SLYKE, R.M. and WETS, R., "L-Shaped Linear Programs with Applications to Optimal Control and Stochastic Programming", O.R. Center 1966-17, University of California, Berkeley, June 1966.

116. VAN SLYKE, R. and WETS, R., "Programming Under Uncertainty and Stochastic Optimal Control", SIAM Journal, Vol. 4, No. 1, 1966.

117. VARAIYA, P.P., "Decomposition of Large-Scale Systems", SIAM Journal, Vol. 4, No. 1, 1966.

118. WAGNER, Harvey, "A Linear Programming Solution to Dynamic Leontief Type Models", Management Science, Vol 3, No. 3, 1957, p. 234-254.

119. WETS, R., "Programming under Uncertainty", 1964, Ph.D. Thesis, University of California, Berkeley.

120. WILDE, D.J., "Production Planning of Large Systems", Chemical Engineering Progress, January 1963.

121. WILLIAMS, J.D., "The Method of Continuous Coefficients, Parts I and II", Report No. ECC 60.3, Socony, 1960.

122. WILLIAMS, A.C., "A Treatment of Transportation Problems by Decomposition", J. Soc. Indust. Appl. Math. Vol. 10, No. 1, March 1962, pp. 35 - 48.

123. WHALEN, "Linear, Programming, Optimal Control, IRE Trans on Auto Control, Vol. AC-7, #4 (July, 1962) 1962 Ph.D. Thesis, University of California, Berkeley.

124. WOLFE, P., "Accelerating the Cutting Plane Method for Non-linear Programming", J. Soc. Indust. Appl. Math., Vol 9, No. 3, September 1961, pp. 481-488.

125. WOLFE, P., and DANTZIG, G.B., "Linear Programming in a Markov Chain", Operations Research 10 (1962), 702-710.

126. WOOD, Marshall, K. and DANTZIG, G.B., "The Programming of Interdependent Activities: General Discussion", Econometrica, Vol 17, No. 3 & 4, July-October 1949, pp. 193-199. Also in Activity Analysis of Production and Allocation, Koopmans, ed., 1951-1, pp. 15-18, and following chapter by Dantzig.

127. ZSCHAU, E.V.W., "A Primal Decomposition Algorithm for Linear Programming", Graduate School of Business, Stanford University, January 1967.

128. ZADEH, L., "Note on Linear Programming and Optimal Control, IRE Trans on Auto. Control, Vol AC-7, #4, July 1962.

Unclassified
Security Classification

**DOCUMENT CONTROL DATA - R&D**
*(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)*

1. ORIGINATING ACTIVITY (Corporate author)
Department of Operations Research
Stanford University
STANFORD, California 94305

2a. REPORT SECURITY CLASSIFICATION
Unclassified

2b. GROUP

3. REPORT TITLE
Large-Scale Linear Programming

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)
Technical Report

5. AUTHOR(S) (Last name, first name, initial)
DANTZIG, George B.

| 6. REPORT DATE | 7a. TOTAL NO. OF PAGES | 7b. NO. OF REFS |
|---|---|---|
| November 1967 | 26 | 128 |

| 8a. CONTRACT OR GRANT NO. | 9a. ORIGINATOR'S REPORT NUMBER(S) |
|---|---|
| N-00014-67-A-0112-0011<br>b. PROJECT NO.<br>NR-047-064<br>c.<br>d. | Technical Report No. 67-8<br>9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) |

10. AVAILABILITY/LIMITATION NOTICES
Distribution of this document is unlimited

| 11. SUPPLEMENTARY NOTES | 12. SPONSORING MILITARY ACTIVITY |
|---|---|
| | Logistics and Mathematical Statistics Branch, Mathematical Sciences Division, Office of Naval Research WASHINGTON, D.C. 20360 |

13. ABSTRACT

From its inception Linear Programming was envisioned as being applied to large detailed dynamic models of economic and industrial systems. Difficulties of obtaining input data, making use of detailed output data, and the cost of computation have in the past limited applications. Three types of approaches have been proposed for efficient computation. These are reviewed in terms of typical matrix structures to which they are applicable. A list of 128 references is appended.

DD FORM 1 JAN 64 1473

Unclassified
Security Classification

Unclassified

Security Classification

| 14. KEY WORDS | LINK A | | LINK B | | LINK C | |
|---|---|---|---|---|---|---|
| | ROLE | WT | ROLE | WT | ROLE | WT |
| Large Scale<br>Linear Programming<br>Decomposition Principle<br>Compact Inverse<br>Staircase Matrices | | | | | | |

**INSTRUCTIONS**

1. **ORIGINATING ACTIVITY:** Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2a. **REPORT SECURITY CLASSIFICATION:** Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. **GROUP:** Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. **REPORT TITLE:** Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. **DESCRIPTIVE NOTES:** If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. **AUTHOR(S):** Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. **REPORT DATE:** Enter the date of the report as day, month, year, or month, year. If more than one date appears on the report, use date of publication.

7a. **TOTAL NUMBER OF PAGES:** The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. **NUMBER OF REFERENCES:** Enter the total number of references cited in the report.

8a. **CONTRACT OR GRANT NUMBER:** If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. **PROJECT NUMBER:** Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. **ORIGINATOR'S REPORT NUMBER(S):** Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. **OTHER REPORT NUMBER(S):** If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. **AVAILABILITY/LIMITATION NOTICES:** Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through _____."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through _____."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through _____."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. **SUPPLEMENTARY NOTES:** Use for additional explanatory notes.

12. **SPONSORING MILITARY ACTIVITY:** Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. **ABSTRACT:** Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS)*, *(S)*, *(C)*, or *(U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. **KEY WORDS:** Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.

DD FORM 1 JAN 64 1473 (BACK)

Unclassified

Security Classification